New Year Mathematical Card
or
V Points Constant

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The article describes an attempt to define a new mathematical constant - the probability of obtaining a hyperbola or an ellipse when throwing five random points on a plane.

Keywords: plane curve of the second order, hyperbola, ellipse, Mathcad, pseudo-random number

The author has such a long pre-New Year's habit. He and his students develop and post on the site of users of the mathematical package Mathcad (https://community.ptc.com/t5/PTC-Mathcad/ct-p/PTCMathcad) [1] greeting animated cards with some entertaining mathematical meaning.

At the end of 2017, a New Year's postcard was published: in a square on a plane, five points are randomly scattered 1000 times, through which a plane curve of the second order was drawn. Anyone who sees on this postcard all seven possible curves (seven is a beautiful number!), Then in the New Year happiness and luck awaits: see https://community.ptc.com/t5/PTC-Mathcad-Questions/New-2018-Year/m-p/495771.

The prehistory of this New Year's postcard is as follows. In one lesson with his students, the author showed a sagging chain (see Figure 1 [3]) and asked a question, the graph of what function this all resembles. The students answered in chorus that it was a parabola. Then the coordinates of five points (left, right, bottom and two points in the middle of the "branches" of the chain) were taken from the photograph of the chain in the environment of the graphic editor Paint, through which a plane curve of the second order was drawn. These five points were taken from the photographs of the sagging chain in different positions and with varying degrees of tension. After processing the data, it turned out that the curve of the second order passing through the five "chain" points was, of course, not a parabola and not even one of the branches of the hyperbola, but ... an arc of an ellipse. The lesson ended with the description of the function of the chain line, which more or less accurately (taking into account measurement errors) passes through three, four, five and more points "taken" from the photograph of the sagging chain.

The question arose and which second-order curve will pass through five points randomly chosen not on the chain image, on the whole plane. Having at hand a computer with the mathematical program Mathcad [4], it can be tried to determine by the method of statistical tests (Monte Carlo) [5].

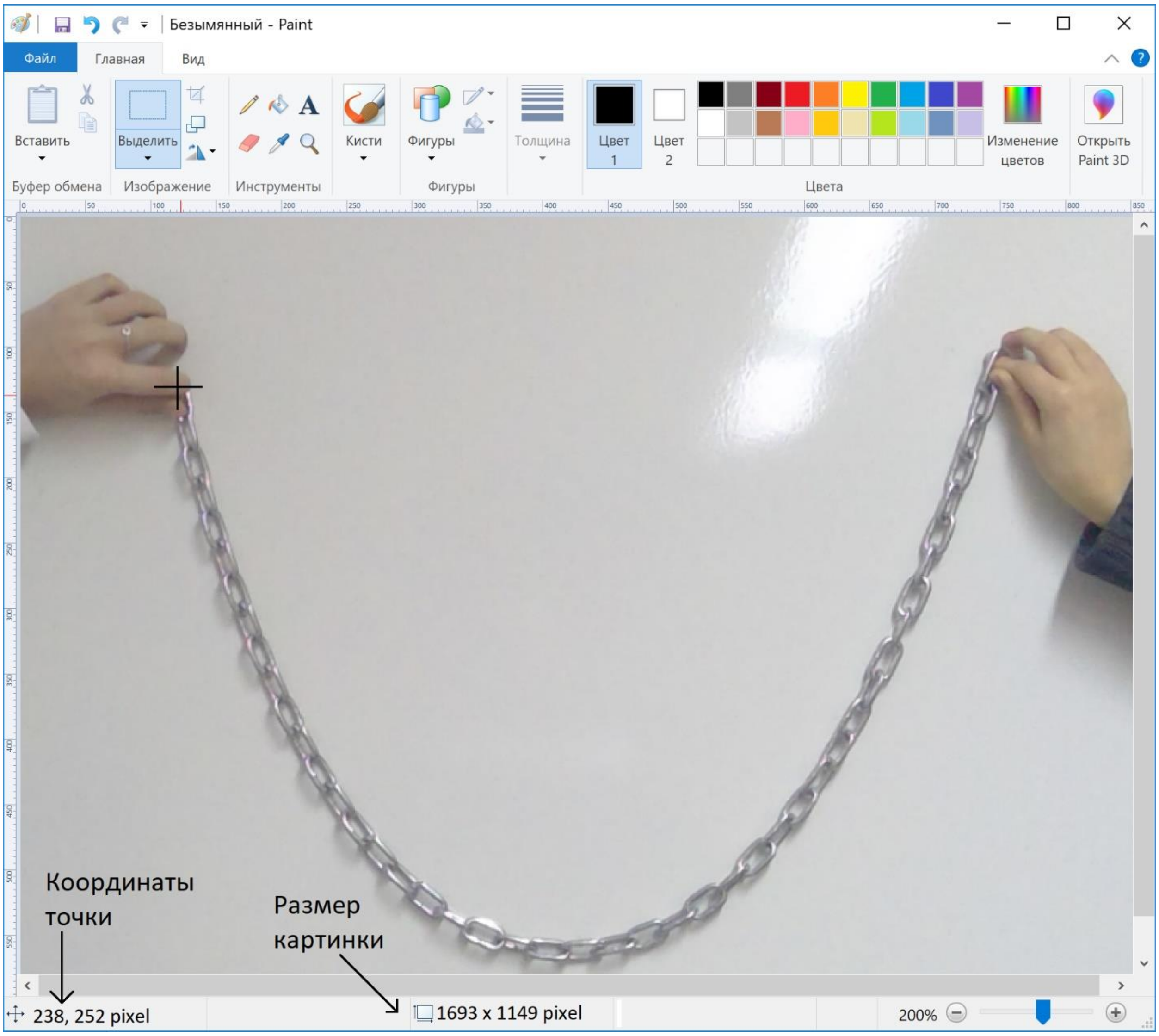

Fig. 1. Experiment with the sagging chain

It is known that through five points on the plane one can draw such curves of the second order [2]:

1. Two branches of the hyperbola.

2. The ellipse.

3. Parabola (transitional case from hyperbola to ellipse).

4. Circle (special case of an ellipse).

5. Two intersecting lines (degenerate two branches of the hyperbola).

6. Two parallel straight lines.

7. One straight line (a special case of cases 5 and 6).

Five randomly discarded points practically (not theoretically) can hold, of course, only two curves: a hyperbola with two branches and an ellipse - see Fig. 2. The other five curves the author and his students "drew" on the New Year's card manually, asking the "correct" coordinates of five points, "not relying on the case."

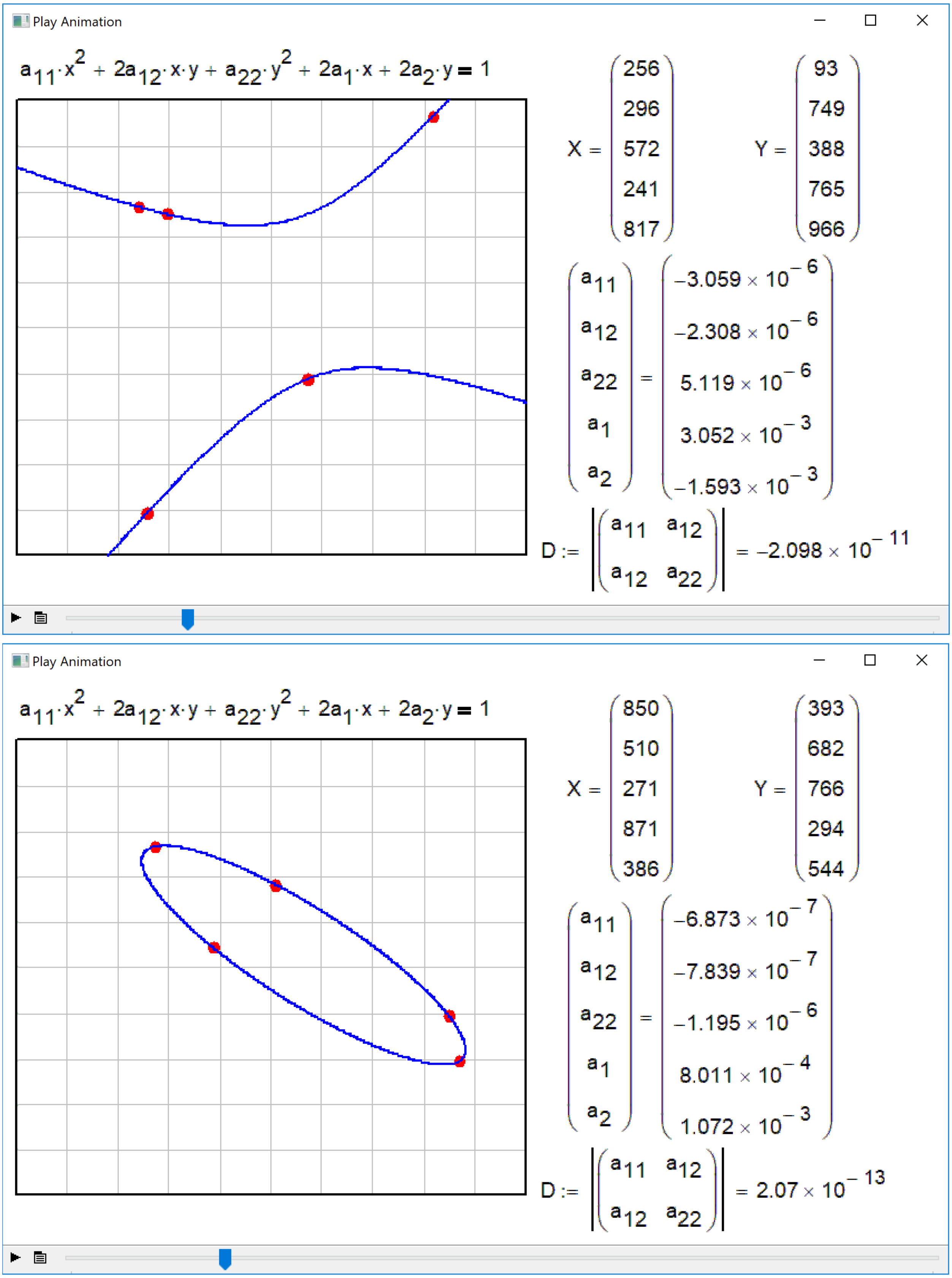

Fig. 2. Hyperbola (above) and an ellipse drawn through five random points

The Fig 2 shows two frames of the New Year's animation card with a hyperbola and an ellipse. The coordinates of the five random points (the values of the vectors X and Y) were also shown on the card, the values of the five coefficients of the second-order curve equation (a_{11}, a_{12}, a_{22}, a_1 and a_2: see the equation on the top of the Fig. 1), as well as the values of the invariant D over which the curve was determined: $D < 0$ is a hyperbola and $D > 0$ is an ellipse [2].

For $D = 0$ (and under other additional conditions), a parabola (the transient case from the ellipse to the hyperbola) must be obtained. But this case, we repeat, took place only with the artificial, and not random assignment of the values of the vectors X and Y.

Those who received this New Year card in addition (for "full happiness" in the coming 2018) were asked to calculate how many times they saw the hyperbola, and how many times - the ellipse.

It turned out that the hyperbola appeared in approximately 71.84% of cases, and the ellipse in the remaining 28.16%. This was calculated, of course, not through viewing animation frames, but through a statistical computer experiment – see on the Fig. 3 the Mathcad program.

$$
(H\ \ E\ \ P) := \left|\begin{array}{l}
\text{for } i \in 1..1000000 \\
\quad \left|\begin{array}{l}
X \leftarrow \text{runif}(5,-1,1) \\
Y \leftarrow \text{runif}(5,-1,1) \\
\begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_1 \\ a_2 \end{pmatrix} \leftarrow \text{lsolve}\left[\begin{pmatrix}
(X_1)^2 & 2\cdot X_1\cdot Y_1 & (Y_1)^2 & 2\cdot X_1 & 2\cdot Y_1 \\
(X_2)^2 & 2\cdot X_2\cdot Y_2 & (Y_2)^2 & 2\cdot X_2 & 2\cdot Y_2 \\
(X_3)^2 & 2\cdot X_3\cdot Y_3 & (Y_3)^2 & 2\cdot X_3 & 2\cdot Y_3 \\
(X_4)^2 & 2\cdot X_4\cdot Y_4 & (Y_4)^2 & 2\cdot X_4 & 2\cdot Y_4 \\
(X_5)^2 & 2\cdot X_5\cdot Y_5 & (Y_5)^2 & 2\cdot X_5 & 2\cdot Y_5
\end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right] \\
D \leftarrow \left|\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\right| \\
\text{if}(D < 0, H \leftarrow H + 1, \text{if}(D > 0, E \leftarrow E + 1, P \leftarrow P + 1))
\end{array}\right. \\
(H\ \ E\ \ P)
\end{array}\right.
$$

$$H = 719484 \qquad E = 280516 \qquad P = 0 \qquad H + E + P = 1000000 \qquad \frac{E}{H} = 0.389885$$

Fig. 3. Calculation of hyperbolas and ellipses, resulting in a square area

The Fig. 3 shows the program for calculating the number of dropped hyperbolas (H) and ellipses (E) when throwing five random points ten million times in a square of 2 by 2. At the same time (just in case!) The number of dropped parabolas (P) was counted.

In the Mathcad document in Fig. 3 it suffices to explain the essence of the following operators and functions:

1. The for comand forms a loop with the parameter i of throwing points into a square.
2. The function runif generates 5 numbers (the first argument of the function) with a random (pseudo-random) distribution in the interval from -1 to 1 (the second and third arguments of the runif function).

3. The lsolve function returns the solution of the system of linear algebraic equations (SLAE), the coefficient matrix for the unknowns is the first argument of the lsolve function, and the vector of the free terms is the second argument. The SLAE solutions are the vector of the coefficients of the required second-order equation (a_{11}, a_{12}, a_{22}, a_1 and a_2).
4. The if function counts the "dropped" hyperbolas, ellipses and parabolas.

At the end of Figure 3, it is shown that, with ten million tossing, the five points of the hyperbola (H) fell 719 484 times, the ellipse (E) was 280 516 times, and parabola (P), as expected, never. The same approximate figures are obtained with other amounts of casts, and with different sizes of the square area, where five points were thrown. We will return to the form of the region below!

Many people, after receiving a greeting card, read the message, admire the picture and ... throw a postcard into a desk drawer or even throw it away. The publication on the Mathcad user forum of the "New Year" postcard with hyperbolas and ellipses had other consequences:

1. It was suggested that a new mathematical constant of 0.2806 was opened ... She was even given a joke or seriously given a name: V Points[1]. One colleague of the author from the department of higher mathematics asserts that he has found a way of analytical, not statistical calculation of this constant, but he does not have time to put it all in mathematical language, since he is now completely immersed in writing a doctoral dissertation[2].

2. One the forum visitor Frank Purcell from Chicago suggested that this constant can be determined (estimated) in another way - when solving the problem of four points (IV points problem[3]), through which two intersecting parabolas are held, which break a square area into certain zones (see please https://community.ptc.com/t5/PTC-Mathcad-Questions/Firecrackers-2018-or-5-th-order-curve-and-20-points/m-p/496511 and https://community.ptc.com/t5/PTC-Mathcad-Questions/Hyperbola-and-Ellipse-new-math-constant/td-p/495992). The fifth random point can fall into one of these zones, which determines what will be built through these five points - the hyperbola or the parabola. The calculation of the sum of the areas of these zones will give our constant. The four-point problem is described on the website http://mathworld.wolfram.com (см. E. Weisstein's article on Sylvester's Four-Point Problem).

3. At the forum of Mathcad users a certain race started - who will throw more points into the square and who will manage to draw through them a curve of more and higher order. Forum users began to squar 9 points (a curve of the third order - a cube), 14 points (4th order), and so on to the curve... 50th order (Werner Exinger) - see Fig. 4 some such curves.

[1] V Points - so computer translators Google translate the Russian author's name В Очков.
[2] This man wanted to become like Pierre Fermat, who wrote down his great theorem, but did not give her proof. And only a few centuries later it was found.
[3] Four points are randomly dashed to the plane and it is determined whether they form a convex quadrangle or one of the points is inside the triangle formed by the other three points.

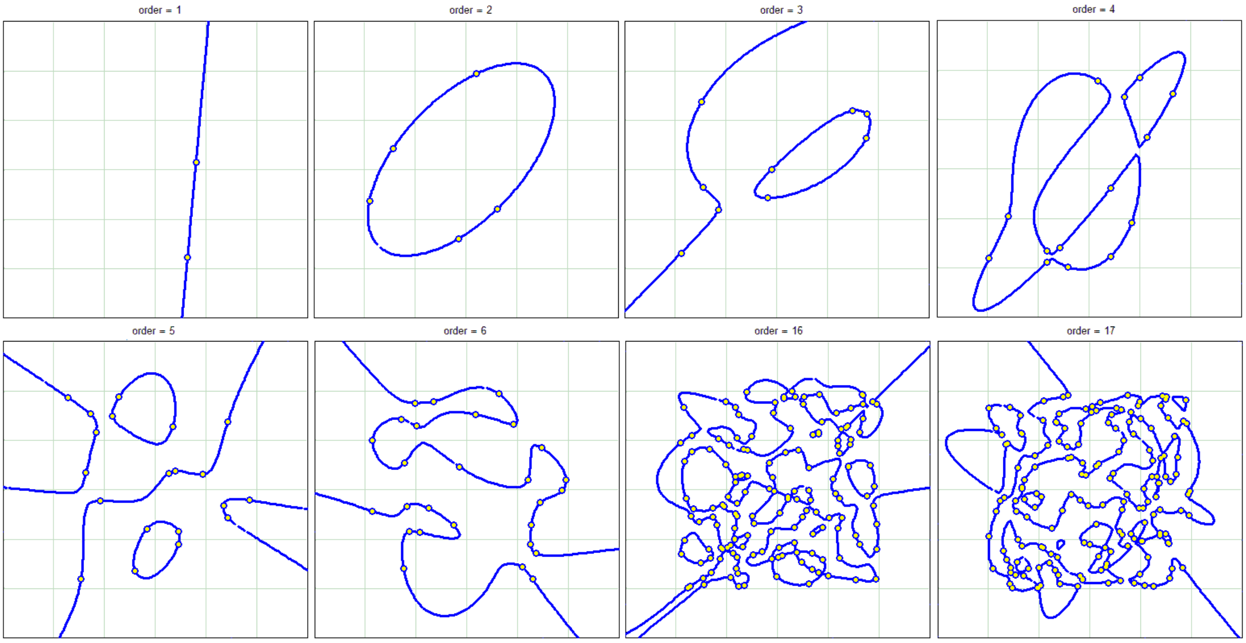

Fig. 4. Curves of different orders passing through different number of points

5. If five points are not squared, but in a circle, then a slightly different value of the constant V Points will be obtained - see Fig. 5. What is it? Error of statistical experiment or proof of absence of V Points constant?

```
N := 10^6        r := 3

(H)     for i ∈ 1..N
(E) :=     while 1
              X ← runif(5, −r, r)
              Y ← runif(5, −r, r)
              F ← 0
              for i ∈ 1..5
                 F ← F + 1  if  (X_i)^2 + (Y_i)^2 > r^2
              break  if  F = 0
           (a11)            [ (X1)^2  2·X1·Y1  (Y1)^2  2·X1  2·Y1 ]   (1)
           (a12)            [ (X2)^2  2·X2·Y2  (Y2)^2  2·X2  2·Y2 ]   (1)
           (a22) ← lsolve(  [ (X3)^2  2·X3·Y3  (Y3)^2  2·X3  2·Y3 ] , (1) )
           (a1 )            [ (X4)^2  2·X4·Y4  (Y4)^2  2·X4  2·Y4 ]   (1)
           (a2 )            [ (X5)^2  2·X5·Y5  (Y5)^2  2·X5  2·Y5 ]   (1)
           D ← | (a11  a12) |
               | (a12  a22) |
           if(D < 0, H ← H + 1, E ← E + 1)
        (H)
        (E)
```

$H = 701742$ $\quad E = 298258$ $\quad \frac{H}{N} = 0.70174$ $\quad \frac{E}{N} = 0.29826$

Fig. 5. Calculation hyperbolas and ellipses in a circular region

By the way, the students (see the beginning of the article) were also asked how they feel: a plane is an infinite continuation of a square or a circle? Again, without thinking about the legality of posing such a question, they answered in chorus that the square. And this is understandable - in school "Cartesian" coordinate system on the plane or in the volume "pass", but almost do not touch the polar, cylindrical and spherical coordinates. This lesson ended with a story about how the function describing the chain line, independently of each other and almost simultaneously discovered Bernoulli, Huygens and Leibniz [6].

Вывод. Сделана попытка открытия новой математической константы, связанной с гиперболой и эллипсом. Предлагаем читателям доказать или опровергнуть существование данной константы, а также попытаться доказать наличие VI points, VII points, VIII points, IX points etc. констант.

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